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Almost sure stability of the thin-walled beam subjected to end moments

Ratko Pavlović Predrag Kozić *

Abstract

The dynamic stability problem of the thin-walled beams subjected to end moments is studied. Each moment consists of constant part and time-dependent stochastic non-white function. Closed form analytical solutions are obtained for simply supported boundary conditions. By using the direct Liapunov method almost sure asymptotic stability condition is obtained as function of stochastic process variance, damping coefficient, geometric and physical parameters of the beam.

The stability regions for I-cross section and narrow rectangular cross section are shown in variance - damping coefficient plane when stochastic part of moment is Gaussian zero-mean process with variance σ^2 and harmonic process with amplitude A.

Keywords: *Random loading, almost sure stability, Liapunov functional, Gaussian and harmonic processes, stability regions, mode number*

*Mechanical Engineering Faculty, Beogradska 14, 18000 Niš, Serbia & Montenegro, e-mail address: ratko@masfak.ni.ac.yu

1 Introduction

Many engineering structures like bridges, cranes and aircraft are made up in some way or another of a number of thin-walled elements. Such structures find wide applications because of the optimal strength and stiffness. Studying of dynamical problems concerning to stability and oscillations is very important. This structures often experience a combination of static and dynamic loads, and there are many situations where the dynamic behavior of the structure depends significantly on the static stress field.

The problem of elastic stability of thin-walled beams, cross-sections with two axes of symmetry, subjected to equal and constant end moments, first was solved by Timoshenko [7].

In the case when thin-walled beam is subjected to an axial load and end moments, Joshi and Suryanayan [3] obtained solution for the coupled flexural-torsional vibration.

Dynamic stability of simply supported thin, elastic beam subjected to stochastic white-noise excitations is considered by Ariaratnam [1]. Applying Galerkin method the problem is reduced to consideration of parametric oscillations of discrete system.

Tylikowski [8], applied direct Liapunov method to uniform stochastic stability analysis of a thin-walled double-tee beams loaded by equal end moments. The intention of the present paper is to investigate almost sure asymptotic stability of thin-walled beams subjected to time-dependent stochastic end moments.

2 Problem formulation

Let us consider the flexural-torsional stability of a homogeneous, isotropic, thin-walled beam with two planes of symmetry. The beam is assumed to be loaded in the plane of greater bending rigidity by two equal couples acting at the ends, (Fig. 1).

The governing differential equations for the coupled flexural and torsional motion of the beam can be written as [2], [8]:

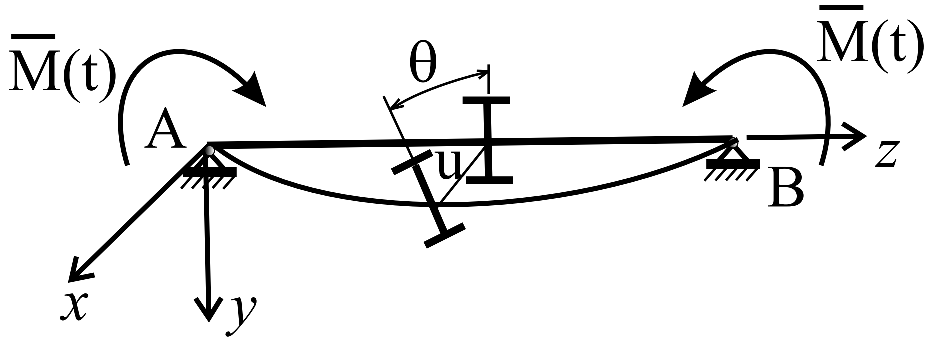

Figure 1: Geometry of thin-walled I-beam subjected to end moments

$$\rho A\frac{\partial^2 V}{\partial\tau^2}+\alpha_V\frac{\partial V}{\partial\tau}+EI_x\frac{\partial^4 V}{\partial Z^4}+\overline{F}(t)\frac{\partial^2 V}{\partial Z^2}+\overline{M}(t)\frac{\partial^2\theta}{\partial Z^2}=0, \qquad (1)$$

$$\rho I_p\frac{\partial^2\theta}{\partial\tau^2}+\alpha_\theta\frac{\partial\theta}{\partial\tau}-\left(GJ-\overline{F}(t)\frac{I_p}{A}\right)\frac{\partial^2\theta}{\partial Z^2}+\overline{M}(t)\frac{\partial^2 V}{\partial Z^2}+EI_s\frac{\partial^4\theta}{\partial Z^4}=0, \quad (2)$$

where U – flexural displacement in the x-direction, θ - torsional displacement, ρ - mass density, A – area of the cross-section of beam, I_x, I_p, I_s – axial, polar and sectorial moment of inertia, J – Saint-Venant torsional constant, E – Young modulus of elasticity, G – shear modulus, α_U, α_θ - viscous damping coefficients, τ - time and Z – axial coordinate.

Using the following transformations:

$$U=u\sqrt{\frac{I_p}{A}}\ ,\quad Z=z\ell\ ,\quad \overline{M}(t)=M_{cr}\left(M_o+M(t)\right)\ ,\quad \tau=k_t t$$
$$F_{cr}=\frac{\pi^2 EI_x}{\ell^2}\ ,\quad M_{cr}=\frac{\pi}{\ell}\sqrt{EI_xGJ}\ ,\quad k_t^2=\frac{\rho A\ell^4}{EI_x}\ ,\quad e=\frac{AI_s}{I_xI_p} \qquad (3)$$
$$\beta_1=\frac{1}{2}\alpha_v\frac{\ell^2}{\sqrt{\rho AEI_x}}\ ,\quad \beta_2=\frac{1}{2}\alpha_\theta\ell^2\sqrt{\frac{A}{\rho EI_xI_p^2}}\ ,\quad S=\frac{GJA\ell^2}{\pi^2EI_xI_p}\ ,$$

where ℓ- length of the beam, M_{cr} - critical bending moment for simply supported beam, S – slenderness parameter, β_1 and β_2 - reduced viscous damping coefficients, we get governing equations as:

$$\frac{\partial^2 u}{\partial t^2} + 2\beta_1 \frac{\partial u}{\partial t} + \frac{\partial^4 u}{\partial z^4} + \pi^2 \sqrt{S} \left(M_o + M(t) \right) \frac{\partial^2 \theta}{\partial z^2} = 0 \tag{4}$$

$$\frac{\partial^2 \theta}{\partial t^2} + 2\beta_2 \frac{\partial \theta}{\partial t} - \pi^2 S \frac{\partial^2 \theta}{\partial z^2} + \pi^2 \sqrt{S} \left(M_o + M(t) \right) \frac{\partial^2 u}{\partial z^2} + e \frac{\partial^4 \theta}{\partial z^4} = 0. \tag{5}$$

Taking free warping displacement and zero angular displacements into account, boundary conditions for simply supported beam are:

$$\begin{aligned} u(t,0) = u(t,1) = \frac{\partial^2 u}{\partial z^2}(t,0) = \frac{\partial^2 u}{\partial z^2}(t,1) = 0 \\ \theta(t,0) = \theta(t,1) = \frac{\partial^2 \theta}{\partial z^2}(t,0) = \frac{\partial^2 \theta}{\partial z^2}(t,1) = 0 \end{aligned} \tag{6}$$

The purpose of the present paper is the investigation of almost sure asymptotic stability of the thin-walled beam subjected to stochastic time-dependent end moments. To estimate perturbed solutions it is necessary to introduce a measure of distance $\| \bullet \|$ of solutions of the Eqs.(4) and (5) with nontrivial initial conditions and the trivial one. The equilibrium state of Eqs.(4) and (5) is said to be almost sure stochastically stable, [4] if:

$$P \left\{ \lim_{t \to \infty} \| w(.,t) \| = 0 \right\} = 1, \tag{7}$$

where $w = col(u, \theta)$.

3 Stability analises

With the purpose of applying the Liapunov method, we can construct the functional by means of the Parks – Pritchard's method [5]. Thus, let we write Eqs.(4) and (5) in formal form $\mathcal{L}w = 0$ where $\mathcal{L}$ is matrix:

$$\mathcal{L} = \begin{bmatrix} \dfrac{\partial^2}{\partial t^2} + 2\beta_1 \dfrac{\partial}{\partial t} + \dfrac{\partial^4}{\partial z^4} & \pi^2 \sqrt{S}\,(M_o + M(t))\,\dfrac{\partial^2}{\partial z^2} \\ \pi^2 \sqrt{S}\,(M_o + M(t))\,\dfrac{\partial^2}{\partial z^2} & \dfrac{\partial^2}{\partial t^2} + 2\beta_2 \dfrac{\partial}{\partial t} - \pi^2 S \dfrac{\partial^2}{\partial z^2} + e \dfrac{\partial^4}{\partial z^4} \end{bmatrix} \tag{8}$$

and introduce linear operator:

$$\mathcal{N} = \begin{bmatrix} 2\dfrac{\partial}{\partial t} + 2\beta_1 & 0 \\ 0 & 2\dfrac{\partial}{\partial t} + 2\beta_2 \end{bmatrix} \tag{9}$$

which is formal derivative of the operator $\mathcal{L}$ with respect to $\dfrac{\partial}{\partial t}$.

Integrating the scalar product of the vectors $\mathcal{L}w$ and $\mathcal{N}w$ on rectangular $C = \Omega \times \Delta = [z : 0 \leq z \leq 1] \times [\tau : 0 \leq \tau \leq t]$ with respect to Eqs.(4) and (5), it is clear:

$$\begin{aligned} 2\int_0^t \int_0^1 \Bigg\{ & \left[\frac{\partial^2 u}{\partial t^2} + 2\beta_1 \frac{\partial u}{\partial t} + \frac{\partial^4 u}{\partial z^4} + \pi^2 \sqrt{S}\,(M_o + M(t))\,\frac{\partial^2 \theta}{\partial z^2} \right] \times \\ & \times \left(\frac{\partial u}{\partial t} + \beta_1 u \right) + \left[\frac{\partial^2 \theta}{\partial t^2} + 2\beta_2 \frac{\partial \theta}{\partial t} - \pi^2 S \frac{\partial^2 \theta}{\partial z^2} + \right. \\ & \left. + \pi^2 \sqrt{S}\,(M_o + M(t))\,\frac{\partial^2 u}{\partial z^2} + e \frac{\partial^4 \theta}{\partial z^4} \right] \left(\frac{\partial \theta}{\partial t} + \beta_2 \theta \right) dz\, d\tau = 0. \end{aligned} \tag{10}$$

After applying the partial integration to Eq.(10) the sum of two integrals may be obtained as:

$$V\,\Big|_0^t - \int_0^t \frac{dV}{dt} = 0, \tag{11}$$

where is:

$$\begin{aligned}\mathbf{V} = \int_0^1 \Big\{ & \left(\tfrac{\partial u}{\partial t} + \beta u\right)^2 + \beta^2 u^2 + \left(\tfrac{\partial^2 u}{\partial z^2}\right)^2 - 2\pi^2\sqrt{S}\, M_o \tfrac{\partial v}{\partial z}\tfrac{\partial \theta}{\partial z} + \\ & \left(\tfrac{\partial \theta}{\partial t} + \beta\theta\right)^2 + \beta_2^2\theta^2 + \pi^2 S \left(\tfrac{\partial \theta}{\partial z}\right)^2 + e\left(\tfrac{\partial^2 \theta}{\partial z^2}\right)^2 \Big\}\, dz.\end{aligned} \tag{12}$$

and

$$\begin{aligned}\frac{d\mathbf{V}}{dt} = -\int_0^1 \Bigg[& 2\beta\left(\frac{\partial u}{\partial t}\right)^2 + 2\beta\left(\frac{\partial^2 u}{\partial z^2}\right)^2 - 4\beta\pi^2\sqrt{S}\, M_o \frac{\partial v}{\partial z}\frac{\partial \theta}{\partial z} \\ & + 2\pi^2\sqrt{S}\, M(t)\frac{\partial^2 \theta}{\partial z^2}\left(\frac{\partial u}{\partial t} + \beta u\right) + 2\beta\left(\frac{\partial \theta}{\partial t}\right)^2 + + 2\beta\pi^2 S\left(\frac{\partial \theta}{\partial z}\right)^2 + \\ & + 2\pi^2\sqrt{S}\, M(t)\frac{\partial^2 u}{\partial z^2}\left(\frac{\partial \theta}{\partial t} + \beta\theta\right) + 2\beta e\left(\frac{\partial^2 \theta}{\partial z^2}\right)^2 \Bigg]\, dz.\end{aligned} \tag{13}$$

For simplicity, it is taken that $\beta_1 = \beta_2 = \beta$.

Functional $\mathbf{V}$ will be Liapunov functional if it is positive definite. Using well known relations:

$$\begin{aligned}&\int_0^1 \left(\tfrac{\partial^2 u}{\partial z^2}\right)^2 dz \geq \pi^2 \int_0^1 \left(\tfrac{\partial u}{\partial z}\right)^2 dz\,, \\ &\int_0^1 \left(\tfrac{\partial^2 \theta}{\partial z^2}\right)^2 dz \geq \pi^2 \int_0^1 \left(\tfrac{\partial \theta}{\partial z}\right)^2 dz.\end{aligned} \tag{14}$$

Omitting dynamical terms, we can write:

$$\mathbf{V} \geq \pi^2 \int_0^1 \left[\left(\frac{\partial u}{\partial z}\right)^2 - 2\sqrt{S}\, M_o \frac{\partial u}{\partial z}\frac{\partial \theta}{\partial z} + (e+S)\left(\frac{\partial \theta}{\partial z}\right)^2 \right] dz \tag{15}$$

Hence, the positive definite condition reduces to relation:

$$M_o^2 \leq 1 + \frac{e}{S} \tag{16}$$

which is equal with Timoshenko's stability condition for thin-walled beam subjected to constant end moments M_0.

Relation (13) can be written in the form:

$$\frac{d\mathbf{V}}{dt} = -2\beta\mathbf{V} + 2\mathbf{U}, \tag{17}$$

where $\mathbf{U}$ is an auxiliary functional defined as:

$$\mathbf{U} = \int_0^1 \left[\left(2\beta^2 u - \pi^2\sqrt{S}M(t)\frac{\partial^2 \theta}{\partial z^2}\right)\left(\frac{\partial u}{\partial t} + \beta u\right) + \right.$$
$$\left. - \left(2\beta^2 \theta - \pi^2\sqrt{S}M(t)\frac{\partial^2 u}{\partial z^2}\right)\left(\frac{\partial \theta}{\partial t} + \beta\theta\right)\right] dz. \tag{18}$$

Now we attempt to construct a bound:

$$\mathbf{U} \leq \lambda\mathbf{V}, \tag{19}$$

where λ is unknown function.

Proceeding similarly as Kozin [4], we have to solve an additional variational problem:

$$\delta(\mathbf{U} - \lambda\mathbf{V}) = 0. \tag{20}$$

By using the associated Euler's equations we obtain:

$$
\begin{gathered}
2\beta^2\left(\frac{\partial u}{\partial t}+\beta u\right)-\pi^2\sqrt{S}M(t)\left(\frac{\partial^3\theta}{\partial t\partial z^2}+2\beta\frac{\partial^2\theta}{\partial z^2}\right)- \\
-2\lambda\left[\beta\left(\frac{\partial u}{\partial t}+\beta u\right)+\frac{\partial^4 u}{\partial z^4}+\pi^2\sqrt{S}M_o\frac{\partial^2\theta}{\partial z^2}\right]=0, \\
2\beta^2\left(\frac{\partial \theta}{\partial t}+\beta \theta\right)-\pi^2\sqrt{S}M(t)\left(\frac{\partial^3 u}{\partial t\partial z^2}+2\beta\frac{\partial^2 u}{\partial z^2}\right)- \\
-2\lambda\left[\beta\left(\frac{\partial \theta}{\partial t}+\beta \theta\right)+e\frac{\partial^4 \theta}{\partial z^4}+\pi^2\sqrt{S}M_o\frac{\partial^2 u}{\partial z^2}-\pi^2 S\frac{\partial^2\theta}{\partial z^2}\right]=0,
\end{gathered} \tag{21}
$$

$$
\begin{gathered}
2\beta^2 u-\pi^2\sqrt{S}M(t)\frac{\partial^2\theta}{\partial z^2}-2\lambda\left(\frac{\partial u}{\partial t}+\beta u\right)=0, \\
2\beta^2 \theta-\pi^2\sqrt{S}M(t)\frac{\partial^2 u}{\partial z^2}-2\lambda\left(\frac{\partial \theta}{\partial t}+\beta \theta\right)=0.
\end{gathered}
$$

After simplifying, we get two equations:

$$
\begin{gathered}
4\beta^4 u-4\beta^2\pi^2\sqrt{S}M(t)\frac{\partial^2\theta}{\partial z^2}+\pi^4 SM(t)\frac{\partial^4 u}{\partial z^4}- \\
-4\lambda^2\left(\beta^2 u+\frac{\partial^4 u}{\partial z^4}+\pi^2\sqrt{S}M_o\frac{\partial^2\theta}{\partial z^2}\right)=0, \\
4\beta^4 \theta-4\beta^2\pi^2\sqrt{S}M(t)\frac{\partial^2 u}{\partial z^2}+\pi^4 SM(t)\frac{\partial^4 \theta}{\partial z^4}- \\
-4\lambda^2\left(\beta^2 \theta+e\frac{\partial^4 \theta}{\partial z^4}+\pi^2\sqrt{S}M_o\frac{\partial^2 u}{\partial z^2}-\pi^2 S\frac{\partial^2\theta}{\partial z^2}\right)=0,
\end{gathered} \tag{22}
$$

Applying linear operators:

$$
\begin{gathered}
L_1=4\beta^4+\pi^4 SM^2_{(t)}\tfrac{\partial^4}{\partial z^4}-4\lambda^2\left(\beta^2-\pi^2 S\tfrac{\partial^2}{\partial z^2}+e^2\tfrac{\partial^4}{\partial z^4}\right), \\
L_2=4\pi^2\sqrt{S}\left(\beta^2 M_{(t)}+\lambda^2 M_o\right)\tfrac{\partial^2}{\partial z^2},
\end{gathered} \tag{23}
$$

we obtain only one equation:

$$\left\{\left[4\beta^4+\pi^4 SM^2_{(t)}\frac{\partial^4}{\partial z^4}-4\lambda^2\left(\beta^2-\pi^2 S\frac{\partial^2}{\partial z^2}+e^2\frac{\partial^4}{\partial z^4}\right)\right]\times\right.$$
$$\times\left[4\beta^4+\pi^4 SM^2_{(t)}\frac{\partial^4}{\partial z^4}-4\lambda^2\left(\beta^2+\frac{\partial^4}{\partial z^4}\right)\right]- \tag{24}$$
$$\left.-16\pi^4 S\left(\beta^2 M_{(t)}+\lambda^2 M_o\right)\frac{\partial^4}{\partial z^4}\right\}u=0.$$

According to the boundary condition (10) we may write the solution in the form:

$$u(z,t)=\sum_{m=1}^{\infty} T_m(t)\sin\alpha_m z \tag{25}$$

and from (21) we obtain algebraic equation of the second order:

$$16A_m\lambda_m^4+4B_m\lambda_m^2+C_m=0, \tag{26}$$

where are:

$$A_m=\left(\beta^2+\alpha_m^4\right)\left(\beta^2+\pi^2 S\alpha_m^2+e\alpha_m^4\right)-\pi^4 SM_o\alpha_m^4,$$
$$B_m=\left(4\beta^4+\pi^4 SM^2_{(t)}\alpha_m^4\right)(2\beta^2+\alpha_m^4+\pi^2 S\alpha_m^2+e\alpha_m^4) \tag{27}$$
$$+8\pi^4 S\beta^2 M_o M(t)\alpha_m^4,$$
$$C_m=\left(4\beta^4-\pi^4 SM^2_{(t)}\alpha_m^4\right)^2.$$

By solving the differential inequality (17), we obtain the following estimation of the functional:

$$V(t)\le V(0)\exp\left[-t\left(\beta-\frac{1}{t}\int_0^t\lambda(\tau)d\tau\right)\right]. \tag{28}$$

Therefore, it can be stated that the trivial solution of equation (**??**) is almost sure asymptotically stable if:

$$\lim_{t\to\infty} \frac{1}{t} \int_0^t \lambda(\tau)d\tau \leq \beta, \tag{29}$$

or, when the process $M(t)$ is ergodic and stationary:

$$E\{\lambda(t)\} \leq \beta, \tag{30}$$

where E denotes the operator of the mathematical expectation, and:

$$\lambda(t) = \lim_{m} \lambda_m(t). \tag{31}$$

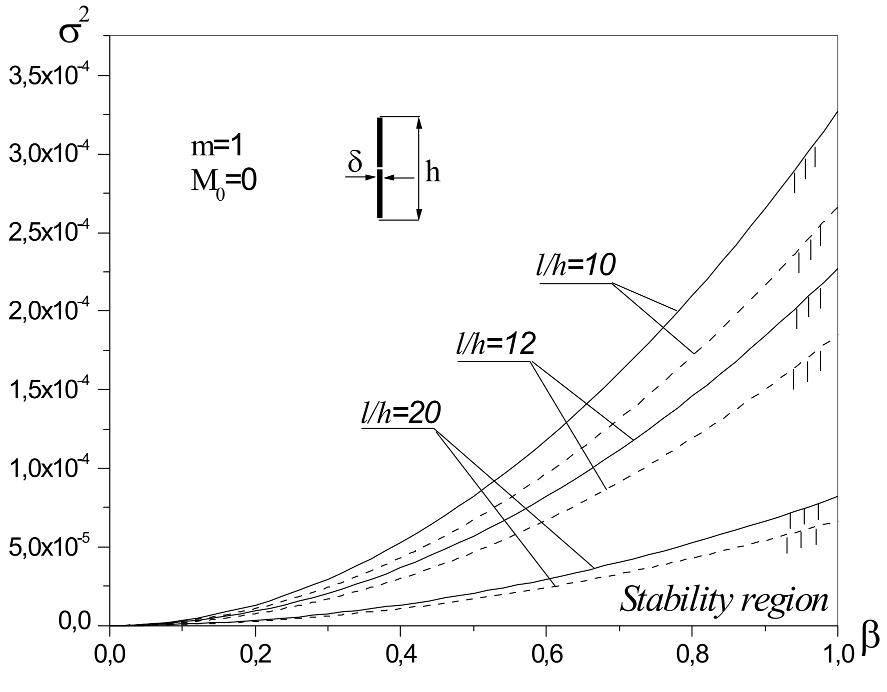

Figure 2: Stability regions for I-section as a function of ℓ/h

4 Numerical results and discussion

The expression (**??**) calculated form (16) and inequalities (19) and (20) give us possibility to obtain critical damping coefficient guaranteeing the almost sure asymptotic stability of the thin-walled beam as a function of statistic characteristics of the end moments. A domain where damping coefficients are greater than the critical damping coefficient is called the almost sure asymptotic stability region. The stability regions as functions of loading variance, damping coefficient, ratio of length to dept of beam cross section, constant component of loading and cross section characteristic are calculated numerically. That calculation is performed for I – section and narrow rectangular cross section.

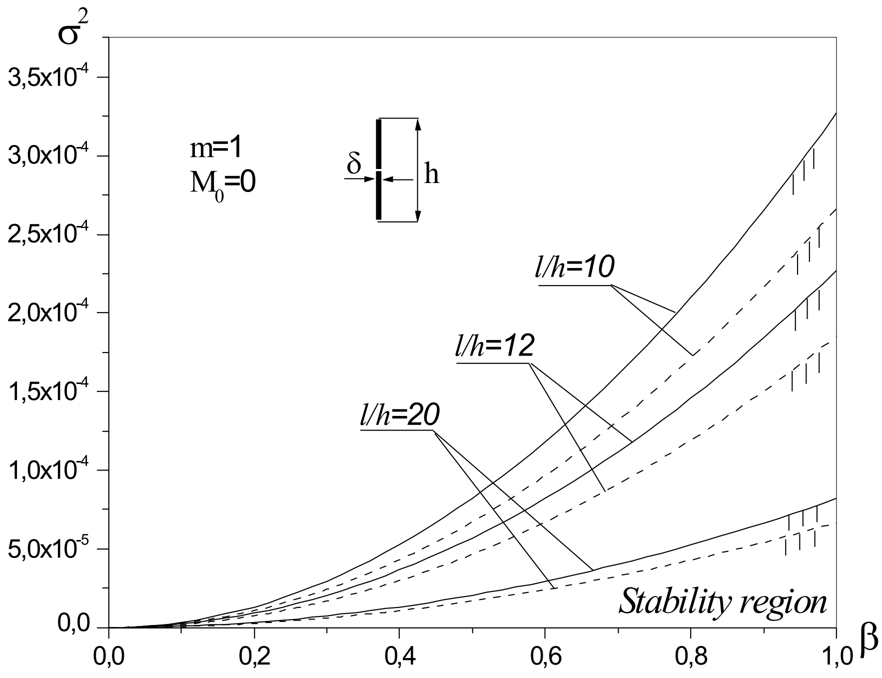

Figure 3: Stability regions for narrow rectangular section as a function of ℓ/h

With respect to standard I–section we can approximately take $h/b \approx 2, b/\delta_1 \approx 11, \delta/\delta_1 \approx 1.5$, where h is depth, b – is width, δ - the thickness of the flanges and δ_1 is thickness of the rib of I – section. These ratios give us $S \approx 0.01928(\ell/h)^2$ and $e \approx 1.276$. For narrow rectangular cross section $S \approx 1.88(\ell/h)^2$ and $e = 0$.

It is well known, if probability density function is given, then greater stability regions are acquired. In this case numerical calculations are performed for the Gaussian zero mean process with variance σ^2 and for the harmonic process with an amplitude A. In order to compare both processes the variance of harmonic process $\sigma^2 = A^2/2$ is used. According to [6] we take the parameters of Gauss-Hermite quadrature for Gaussian process, and Gauss-Chebyshev for harmonic process, where the hatched side of the curves indicates the stable region.

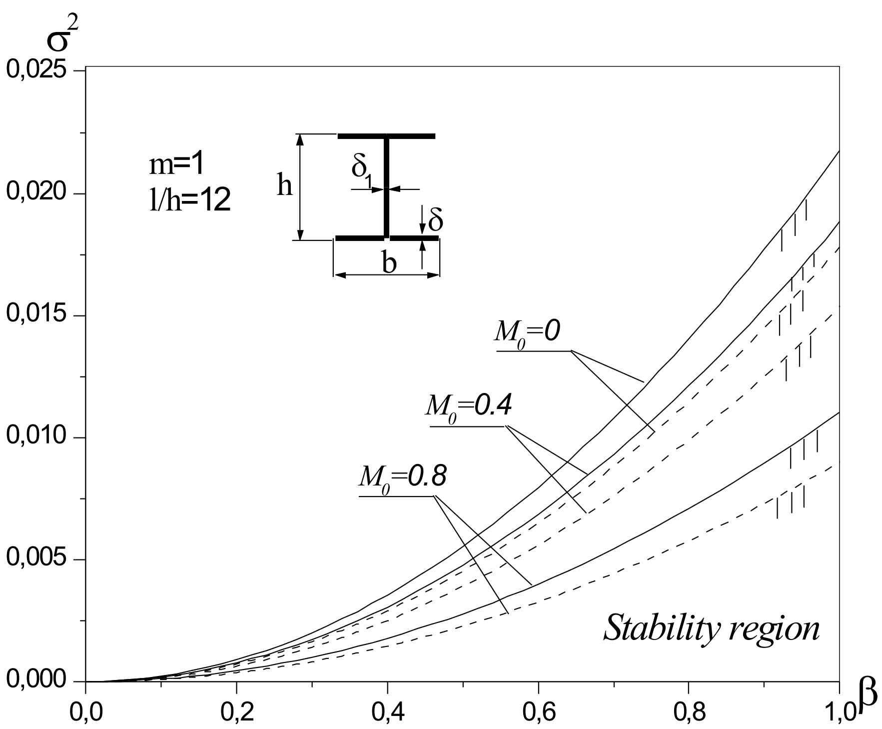

Figure 4: Stability regions for I–section as a function of deterministic component of end the moment

In Fig. 2 and 3 almost sure asymptotic stability regions as functions of ratio of length to depth of cross-section beam ($\ell/h = 10, 12$ and 20 are plotted. The stability regions for I – section, (Fig. 2) are greater then for narrow rectangular cross section, (Fig. 3). Also, we may conclude that when ratio ℓ/h grows, stability regions decrease.

In Fig. 4 the stability regions as functions of constant component of end moments M_0 are given for I–section. The approaching of M_0 to critical value $M_0 = 1$, leads to decreasing of almost sure stability regions.

In Fig. 5 stability regions as function of mode number indicate that higher modes ($m = 2, 3$) are closed to basic mode ($m = 1$) for both processes.

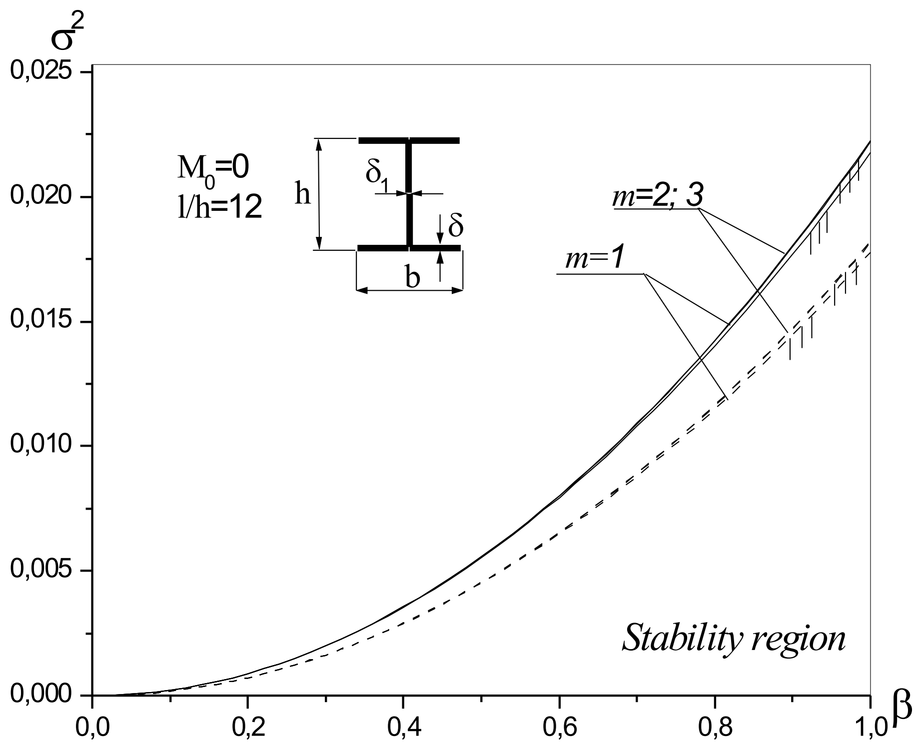

Figure 5: Stability regions for I-section as a function of mode number

Acknowledgements

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Skoro sigurna stabilnost tankozidne grede izložene momentima na krajevima

UDK 534.16

U radu se proučava se problem dinamičke stabilnosti tan- kozidne grede izložene momentima na krajevima. Svaki od momenata se sastoji od konstantnog dela i vremenski pro- menljivog dela sa stohastičkom funkcijom koja nije beli šum. Eksplicitna analitička rešenja su dobijena za slučaj graničnih uslova kada su krajevi prosto oslonjeni. Korišćenjem direktnog metoda Ljapunova dobjen je uslov skoro sigurne asimptotske stabilnosti kao funkcija varijanse stohastičkog procesa, koeficijenta prigušenja, geometrijskih i fizičkih para- metara grede.

Oblasti stabilnosti za I-poprečni presek i uski pravougaoni poprečni presek su prikazani u ravni varijansa – koeficijent prigušenja kada je stohastički deo momenta Gausovski proces sa nultom srednjom vrednošću i varijansom σ^2, a harmonijski proces je sa amplitudom A.