

Recent advances in M^3 (mechanics on the material manifold)

Gérard A. Maugin*

Abstract

After a brief critical review of the basic arguments involved in the various interpretations of the mechanics on the material manifold and a short remark on the relationship of that approach with the late R.Stojanovic's works, some of the recent advances in that field of continuum physics are presented. These include direct consequences in the numerics of solid-mechanics problems, general applications to the theories of plasticity, growth and mixtures, and a nonstandard application to the progress of phase-transition fronts in solids.

1 General remarks by way of introduction

Any new fashionable theoretical development which aims at some generality is plagued by two ingredients. One is the multiplicity of notations introduced for the same objects by different authors. This can be remedied by use of some good "translators". The second is more obnoxious as it relates to questions of priority and interpretations. Although priority matters can sooner or later be discussed and settled in some peaceful atmosphere (see, however, the Leibniz-Newton controversy about calculus), the problem of interpretations remains crucial and usually very hot (see the interpretations of quantum mechanics) while all participants

*Laboratoire de Modélisation en Mécanique, UMR 7607 Université Pierre et Marie Curie Case 162, 4 place Jussieu, 75252 Paris Cedex 05 France (e-mail: gam@ccr.jussieu.fr)

may agree on practical aspects of the matter. The situation appears to be of this kind in the field of continuum physics called the theory of *configurational forces or material forces*. First there are those, like the author, who claim that the theory of material forces is intimately related to a basic invariance of mathematical physics, the homogeneity or lack of homogeneity of the material [1]. This is not *per se* a new law of physics, but only the expression of the projection, onto the material manifold M^3 , of a well known equation of balance of physics, that of linear momentum. All applications, known so far, can be related either to the local fully material expression of this equation when continuous fields are involved, or to the associated jump relation when discontinuous fields are involved across a singular surface, or else to an associated singular global integral when some field singularity is involved [2]-[3]. This unity and its broad range of applications contribute to both the practical strength and the irresistible beauty of this interpretation. Some strong minds, however, want to wrongly attach to this a mystic [4] which they attribute to others while these "others" are fully pragmatic. The equations in question are just canonical projections onto the material manifold of well known equations. It happens that such projections facilitate the appearance and view of some effects that were invisible on the classical writing of the equation (just like taking a photograph from behind a tree obviously shows the hidden face of that tree - the Moon does equally well as an example for that). A material force in that sense is less mystical than the force of gravity - see the first chapter of my book [1]. We can say that this viewpoint is certainly the one accepted by many prominent authors in the field (Rogula [5], Herrmann and Kienzler [3], Maugin, and Trimarco [6], Steinmann [7], etc). Another viewpoint is advocated by Gurtin [8] and some of his co-workers, Podio-Guidugli [9] and Fried [10]. It consists in viewing the balance of canonical momentum as a basic principle of physics, independently of the principle of balance of linear momentum, in a sense a new physical law ! These authors are particularly interested in the treatment of singular surfaces. Why cannot we agree with this?

First, there is no need to introduce a new law when everything is still explainable by the known ones. This is Occam's razor and Mach's idea of scientific economy in a nutshell. Second, like any modern physicist we believe that every conservation law is associated with an invariance

principle. In the present context, the classical balance of linear momentum (with components in *physical* space) is related to the invariance under space translations in that space (invariance under *spatial* translations) in the absence of body forces (e.g., gravity which acts at the current physical point or *placement* x). This is called *homogeneity of space* (Landau and Lifshitz, [11]). Now, material space, i.e., matter per se, has no reason to be homogeneous. On the contrary, many materials are inhomogeneous in a smooth or more irregular or singular way. It is the canonical *material* balance of linear momentum which is generated by this invariance or lack of invariance (invariance under translations on the material manifold!) An ambiguity arises here because the actual placement x of a deformable material is in physical space while the usual space parametrization is the material point \mathbf{X} . The two are related by a time-parametrized mapping, the "motion" of the material. That is why one equation can be deduced from the other for *pure* mechanics and sufficiently smooth fields. This is not the case when the material body possesses other properties [2] and other degrees of freedom [12] which are also parametrized by the material point and time. In that case the balance of canonical momentum captures all degrees of freedom, just in the same way as does the balance of energy. Accordingly, the balance of canonical momentum does not simply relate to a single degree of freedom but to them all *simultaneously*, a fact lightheartedly discarded by those not aware of the clear distinction between fields and parameters in a field theory. It is a pity that respectable authors in mechanics and thermodynamics ignore this elementary fact. This is what led them to blatantly erroneous statements. This is apparently what happened to some authors, otherwise talented analysts, who claim not to be able to understand the deduction of the local balance of canonical momentum in some cases (e.g., electromagnetics bodies - see James [13] versus Fomethé and Maugin [14]) where the proof is laborious but crystal-clear and easily repeatable.

Now what happens in a rigid body or in small deformations (when usually one does not introduce material coordinates)? The material in a rigid body is still subjected to the balance of canonical momentum since its properties may vary from one material point to another. In the deformable case, motion must be treated just like another field, and no more despite the fact that classical motion has the same geometri-

cal and physical dimension as the spatial parametrization (in 3d). In small strains, one must be especially careful to distinguish between the placement and the coordinate frame as otherwise even the best authors may make errors of interpretation (see, Peierls [15]). People working in fluids only are particularly prone to making erroneous interpretations of equations related to material inhomogeneity (see the discussion by Maugin and Trimarco [16] concerning the case of liquid crystals where it is argued that even the most deeply involved scientists - Eshelby himself, Ericksen and Kroener - got confused in this instance).

In the case of a deformable solid exhibiting a singular surface such as a phase-transition front or a classical shock wave, it is clear that the material is not the same (or is *not* in the same phase or state) on both sides of the discontinuity surface which, therefore, breaks the translational symmetry on the material manifold. Accordingly, this dictates that the (co-vectorial) equation governing the change of material (phase) at the interface is none other than the jump relation associated with the balance of canonical momentum. It is such a method of typically mathematical-physics gesture than we used in all cases to construct with success the relevant equation and not an equation issued from some new physical law as Gurtin would like us to believe. We have tried to explain this in detail in our review [2] but apparently in vain so narrow views have some philistines untrained in mathematical physics, not to speak of those, with bad faith, who want to negate this approach on the simple example of a phase transition front. Yes, we know that Eshelby stress is the enthalpy stress tensor, a fact known since Bowen's theory of mixtures [17] and Grinfeld's works [18] - see our comments in [2]. But what counts above all is the wide landscape covered by the general concepts and more inclusive notions such as the generation of the Eshelby stress tensor by means of local structural rearrangements (Epstein and Maugin [19], [20],[21]). In order to substantiate our view we complement a paper already published in this journal [22] with a brief presentation of a series of new developments that easily comfort the wellfoundedness of our approach.

2 Relationship to R.Stojanovic's works

In the 1960s-1970s, nobody had the idea to connect the full projection of continuum mechanics onto the material manifold and the theory of defects, save perhaps for Zorawski who, in the same volume [23], speaks about dislocations and the energy-momentum tensor, but without relationship of the present type. Our comment also applies to W.Noll [24] who introduced powerful ingredients (uniformity maps, connection) but without the further logical step to a full projection of the Cauchy equation of motion or equilibrium onto the material manifold M^3 . The decisive step was taken by Epstein and the author [19] who established that in a continuously *inhomogeneous* elastic body the bulk equilibrium equation projected onto M^3 could be written as

$$\operatorname{div}_R \mathbf{b} = -\mathbf{f}^{inh} := \mathbf{b}:\Gamma \quad (1)$$

in the absence of given body force. Here the divergence operator is the referential one, \mathbf{b} is the quasi-static material Eshleby stress and Γ is the material connection based on the uniformity map \mathbf{K} so that, according to these authors

$$\mathbf{b} = -\frac{\partial \bar{W}}{\partial \mathbf{K}} \mathbf{K}^T \quad , \quad \Gamma = \mathbf{K}^{-1} \cdot (\nabla_R \mathbf{K})^T \quad (2)$$

or, more classically,

$$\mathbf{b} = W \mathbf{1}_R - \mathbf{T}:\mathbf{F}, \quad W = W(\mathbf{F};\mathbf{X}) = \bar{W}(\mathbf{F};\mathbf{K}(\mathbf{X})), \quad \mathbf{T} = \frac{\partial W}{\partial \mathbf{F}}, \quad (3)$$

where W is the elastic energy per unit referential volume and \mathbf{T} is the first Piola-Kirchhoff stress. Whenever the skewsymmetric part of Γ is interpreted as the dislocation density tensor of a *continuously dislocated elastic body* in the manner of E.Kroener and others, then eqn.(1) is the equation that relates the notion of dislocation density and the true material mechanics. R.Stojanovic was very much concerned with the relationship between dislocation processes and the generalized mechanics of continua [25]. In our opinion, the solution of this, especially for *disclinations*, also stems from a consideration of generalized continuum mechanics expressed on the material manifold (compare [26], [27]), something that was clearly out of reach in the times before the untimely death of R.Stojanovic in 1977.

3 Body material forces can be computed and are useful!

In the absence of material inhomogeneities and at any regular material point, eqn. (1) reduces to

$$\operatorname{div}_R \mathbf{b} = \mathbf{0} \quad (4)$$

a pure identity deduced from the classical equilibrium equation (with components in physical space in the present configuration)

$$\operatorname{div}_R \mathbf{T} = \mathbf{0}. \quad (5)$$

Equation (4) is trivially deduced from (5) by right-composition of (5) with \mathbf{F} and integration by parts on account of the elastic energy dependency on \mathbf{F} , and only on \mathbf{F} . A boundary-value problem of nonlinear elasticity is solved on the basis of (5) accompanied by appropriate boundary conditions at the boundary of the material volume. Once eqn.(1) is solved numerically by, e.g., a finite-element scheme, eqn.(4) should be checked. In most cases this is not exactly checked and, as rewritten in a continuous framework, one shall obtain an equation (4) with a source term, i.e.,

$$\operatorname{div}_R \mathbf{b} = -\mathbf{f}^{comp}, \quad (6)$$

where \mathbf{f}^{comp} is a spurious "material force" induced by an incorrect meshing to the problem (this quantity in fact is computed at the nodes of the FEM scheme). It provides a *directional* (since it is a co-vector on M^3) *indicator*, in direction and intensity, of how the FEM nodes should be moved to obtain a better approximation. This was first noticed by Braun [28] and these ideas pursued and put to practice by Maugin [29], Mueller *et al* [30], and Steinmann [31]. Moreover, this practice having shown how easy it is to evaluate these material forces, the computation of such quantities as the *suction force* [32] or *driving force* on the tip of a crack in general physical circumstances has become a rather easy matter. That is, the co-vectorial driving force (not a simple scalar such as the *J-integral*) can be evaluated numerically on account of the numerical solution of a complete problem, not only of homogeneous elasticity, but also in the presence of thermal, anelastic and smooth or abrupt inhomogeneity effects. The quasi-static non-collinear progress of a crack being

attracted by an inhomogeneity using a simple kinetic law of progress has thus been spectacularly illustrated by Mueller and Maugin [33]. Steinmann and co-workers (2002, unpublished) have shown the practicality of such computations in more general thermal-anelastic cases. All these computations are based on a volume integration of the balance of material momentum (generalization of eqn.(1)) and the energy equation, the first of which reading then (cf. Maugin, [12])

$$\operatorname{div}_R \mathbf{b} = - (\mathbf{f}^{inh} + \mathbf{f}^{th} + \mathbf{f}^{anel} + \mathbf{f}^{inert}), \quad (7)$$

where, in the right-hand-side using a revealing notation, we have contributions from true material inhomogeneities, as well as thermal, anelastic, and inertial effects, all in the form of "pseudo-inhomogeneity" material forces. All these may be computed at each node and at each time step, although perhaps not so simply. Note that the volume of integration for (7) evolves with time due to the irreversible progress of the crack in the material body and this is what brings in the picture, by duality, the notion of driving force acting on the tip of the crack.

4 Applications to plasticity, growth and mixtures

Among the most recent advances in the material mechanics of materials, we should emphasize the applications of some of its concepts (in particular, that of uniformity map or local structural rearrangement \mathbf{K} -see eqns.(4)-) to various complex behaviors of matter. It was first realized by Epstein and the author [34] that the Eshelby stress tensor could be viewed as the (bulk) driving force behind finite-strain plasticity. This was facilitated by the identification of the second part of the material tensor \mathbf{b} , as the Mandel stress $\mathbf{M} = \mathbf{T} \cdot \mathbf{F}$ of finite-strain anelasticity, perhaps to be considered in the so-called intermediate configuration. In this case \mathbf{K} is none other than the inverse of the plastic deformation component \mathbf{F}^p of the celebrated multiplicative decomposition $\mathbf{F} = \mathbf{F}^e \cdot \mathbf{F}^p$. No wonder then that the Eshelby stress \mathbf{b} may also be used as the resolved shear stress activating dislocation systems (Kh. Chau Le [35]) Since material growth is very much akin to plasticity save, obviously, for the bulk behavior - i.e., it is a special kind of local structural rearrangement

in which more matter of the same kind is squeezed in at each material point-, it was natural to develop a "material" theory of material growth on the same bases [36], emphasizing the role of the Eshelby stress in such processes. This receives applications in the biomechanics of soft tissues. Finally, with a view to study the biomechanics of objects like remodelled bones it was necessary to envisage a theory of binary mixtures including both a solid with pores and a fluid saturating the latter. Since the material view is essentially one where the most relevant kinematics is that of the inverse motion $\mathbf{X} = \chi^{-1}(\mathbf{x}, t)$ instead of the traditional "direct" motion $\mathbf{x} = \chi(\mathbf{X}, t)$ - this is clearly seen on the fact that equations such as (1), (4) or (7) are directly generated by a change in "particle" \mathbf{X} on the material manifold M^3 rather than by a change in placement \mathbf{x} - which is the case of equations such as (5)-the question naturally arises of the choice of which material manifold should the motions of several co-existing media be pulled back to; the answer is simple and logical, both solid and fluid deformations are pulled back to the *solid* material manifold, i.e., on the skeleton in terms of porous-media mechanics. This is exemplified by the works of S.Quiligotti [37].

5 Application to the progress of phase-transition fronts

To conclude our "experimental" proof of the large field of applications of materials mechanics and material forces, we mention the possibility to use this formulation in the numerics of the progress phase-transition fronts, but in a **non-standard way**. The "standard way" consists in complementing the system of field equations valid in any continuous phase and the system of jump (or transition) relations at the phase front Σ - a special type of shock waves - by a *kinetic relation* relating the local normal propagation velocity of the wave front to the computed driving force at the same singular point. This follows works by Truskinowsky, Abeyaratne, Knowles and others. But we have shown [12] within the logics of viewing the presence of the front as a *breaking of translational symmetry* on the material manifold that the surface driving force \mathbf{f}_Σ was generally a material co-vector computed from the jump of the quasi-static Eshelby stress across the front. Again, this in

general is a *directional indicator* of the further progress of the front, which progress should not contradict the second law of thermodynamics across the front. We have shown that, in fact, for homothermal fronts, the local dissipation at the front is such that

$$\Phi_\Sigma = \mathbf{f}_\Sigma \cdot \bar{\mathbf{V}} \geq 0 \tag{8}$$

The usual pure normal-growth kinetic law then is a generally nonlinear relationship

$$V_N := \bar{\mathbf{V}} \cdot \mathbf{N} = f(\mathbf{N} \cdot \mathbf{f}_\Sigma) \tag{9}$$

where \mathbf{N} is the unit normal to Σ . Such a relation must be provided by a lower-level description essentially one where the front Σ has a non-zero thickness across which dissipative processes take place. Then the progress of the transition front could be treated in an incremental way, using a finite-element scheme in space. The "non-standard way" devised by the author and A. Berezovski [38] is based on the initial remark that the *finite-volume element method* (for short FVM) - favored in many problems of fluid dynamics - considers at once all equations of the continuum problem as balance laws over each cell-volume element. Furthermore, in its discretization it happens to become identical to the thermodynamics of so-called *Schottky* (discrete) *thermodynamical systems*, the exchanges with neighboring system being readily identified to the fluxes at the boundaries of the cells, which boundaries are simple flat surfaces. Accordingly, a problem of phase-transition front propagation is numerically treated in this way, the additional co-vectorial balance equation provided by the balance of canonical momentum (7) in which the associated flux is none other than the Eshelby stress, being logically incorporated in the scheme and the change of the components of the Eshelby stress at the phase change being duly accounted for.

6 Conclusion

We believe that the richness demonstrated by the above briefly reviewed applications should be enough to convince the less imaginative opponent of the inclusive value of the "material mechanics on the material manifold", both from a unifying theoretical viewpoint and from the practical necessity of useful and powerful numerical applications.

Note:

The author has very nice remembrances of the extremely vivid course he took with R.Stojanovic at the CISM, Udine, 1970. This contribution is dedicated to the memory of the latter. GAM benefits from a Max Planck Award for International Co-operation (2001-2005).

References

- [1] G.A.Maugin, *Material Inhomogeneities in Elasticity*, Chapman and Hall, London, 1993.
- [2] G.A.Maugin, *Material Forces: Concepts and Applications*, *Appl.Mech.Rev.*, 48 (1995), 213-245.
- [3] R.Kienzler and G. Herrmann, *Mechanics in Material Space with Applications to Defect and Fracture Mechanics*, Springer-Verlag, Berlin, 2000.
- [4] I.Mueller, *Eshelby Tensor and Phase Equilibrium*, *Theoretical and Applied Mechanics (Belgrade)*, 25 (1999), 77-99 .
- [5] D.Rogula, *Forces in Material Space*, *Arch.Mechanics (Poland)*, 29 (1977), 705-715.
- [6] G.A.Maugin and C.Trimarco, *Elements of Field Theory in Inhomogeneous and Defective Materials* , In : *Configurational Mechanics of Materials*, eds. R.Kienzler and G.A.Maugin, Springer-Verlag , Wien (2001), 55-171.
- [7] P.Steinmann, *On Spatial and Material Settings in Hyperelastodynamics*, *Acta Mechanica* , 156 (2002). 193-218
- [8] M.E.Gurtin, *Configurational Forces as Basic Concepts of Continuum Physics*, Springer-Verlag, New York, 1999.
- [9] P.Podio-Guidugli, 2001, *Configurational balances via variational arguments*, *Interfaces and Free Boundaries*, 3 (2001), 223-232.

- [10] M.E.Gurtin and E.Fried, A Nonequilibrium theory of epitaxial growth that accounts for surface stress and diffusion, Contribution to the Truesdell Memorial Symposium at 14th USNCTAM, Blacksburg, Va, June 2002 (to be published in *J.Elasticity*).
- [11] L.D Landau and E.M.Lifshitz, *Theory of Fields*, Pergamon, Oxford, 1986.
- [12] G.A.Maugin, Thermomechanics of Inhomogeneous-heterogeneous Systems : Applications to the Irreversible Progress of Two and Three-dimensional Defects, *ARI* , 50 (1997), 41-56.
- [13] R.D.James, Configurational Forces in Magnetism with Application to the Dynamics of a Small-scale Ferromagnetic Shape Memory Cantilever, *Continuum Mech.Thermodyn.*, 14 (2002), 55-86.
- [14] A.Fomèthe and G.A. Maugin, Material Forces in Thermoelastic Ferromagnets, *Continuum Mech. Thermodyn.*, 8 (1996), 275-292.
- [15] R.Peierls, *Surprises in Physics*, Princeton University Press, Princeton, N.J., 1979.
- [16] G.A.Maugin and C.Trimarco, On Material and Physical Forces in Liquid Crystals, *Int.J .Engng.Sci.*, 33 (1995), 1663-1678.
- [17] R.Bowen, Toward a Thermodynamics and Mechanics of Mixtures, *Arch.Rat. Mech.Anal.*, 24 (1967), 370-403.
- [18] M.Grinfeld, *Thermodynamic Methods in the Theory of Heterogeneous Systems*, ISIMM Series, Longman, Harlow, Essex, 1991.
- [19] M.Epstein and G.A.Maugin, The Energy-momentum Tensor and Material Uniformity in Finite Elasticity, *Acta Mechanica*, 83 (1990), 127-133.
- [20] M.Epstein and G.A.Maugin, Notions of Material Uniformity and Homogeneity (Opening Lecture of MS1, ICTAM, Kyoto, 1996), In : *Theoretical and Applied Mechanics*, eds. T.Tatsumi, E.Watanabe, and T.Kambe, Elsevier, Amsterdam (1997), 201-215.

- [21] G.A.Maugin, Thermomechanics of Local Structural Rearrangements, In : Elasticity and Anelasticity (90th Anniv.of A.A.Ilyushin), eds. I.Kiyko, M.Israelov and G.Brovko, pp.Moscow University Press, Moscow (2001), 25-35.
- [22] G.A.Maugin, Material Mechanics of Materials (Lecture at National Yugoslav Congress of Mechanics, Belgrade, 2001), Theoretical and Applied Mechanics, 27 (2002) 1-12.
- [23] M.Zorawski , Théorie mathématique des dislocations, Dunod, Paris ,1967.
- [24] W.Noll, Materially Uniform Simple Bodies with Inhomogeneities, Arch.Rat.Mech.Anal., 27 (1967), 1-32.
- [25] R.Stojanovic, Dislocations in the generalized elastic Cosserat continuum, In Proc.IUTAM Symposium Freudenstadt, 1967 , ed. E.Kroener, Springer-Verlag, Berlin (1968) 152-155.
- [26] G.A.Maugin, From Piola's Manifold to Cosserats' Structure, In : Geometry, Continua and Microstructure (Proc.GCM1, Paris,1997), ed.G.A.Maugin, Collection Mathématique "Travaux en cours", Hermann Ed., Paris (1999), 117-120.
- [27] G.A.Maugin, On the Structure of the Theory of Polar Elasticity, Phil.Trans.Roy.Soc.Lond., A356 (1998), 1367-1395.
- [28] M. Braun, Configurational Forces Induced by Finite-element Discretization, Proc.Est.Acad.sci.Math.phys., 46 (1997), 24-31.
- [29] G.A.Maugin, Geometry of Material Space : Its Consequences in Modern Numerical Means , Technische Mechanik (Proc.GMC3, Bad-Herrenalb,1999), 20 (2000), 95-104.
- [30] R.Mueller, S.Kolling and D. Gross D., On Configurational Forces in the Context of the Finite-Element Method, Int.J.Num.Meth.Engng., 53(2002), 1557-1574.
- [31] P.Steinmann, to be published.

- [32] P.Casal, Interpretation of the Rice Integral in Continuum Mechanics, Lett.Appl .Engng.Sci., 16 (1978), 335-348.
- [33] R.Mueller and G.A.Maugin, On Material Forces and Finite Element Discretizations. Computational Mechanics , 29 (2002), 52-60.
- [34] M.Epstein and G.A.Maugin, On the Geometrical Material Structure of Anelasticity, Acta Mechanica, 115 (1995), 119-131 .
- [35] K.C.Le, Thermodynamically Based Constitutive Equations for Single Crystals, In : Geometry, Continua and Microstructure (Proc.GCM1, Paris, 1997), Ed. G.A.Maugin, Collection Mathématique "Travaux en cours", Hermann Editeurs, Paris, (1999), 87-97.
- [36] S.Quiligotti and G.A.Maugin, Eshelbian Mechanics of Binary Solid-Fluid Mixtures. Acta Mechanica, (2002, in press) ; see also contribution by S.Quiligotti in these proceedings.
- [37] A.Berezovski and G.A.Maugin, Thermoelasticity of Inhomogeneous Solids and Finite-volume Dynamic Computations, In : Contributions to Continuum Theories (K.Wilmanski's Festschrift), ed. B.Albers , W.I.A.S. Berlin (2000), 166-173.

Najnoviji rezultati u M^3 (mehanici na materijalnoj mnogostrukosti)

UDK 531.01, 536.76, 539.374, 539.421

Posle kratkog kritičkog pregleda osnovnih argumenata uključenih u različite interpretacije mehanike na materijalnoj mnogostrukosti i jedne kratke primedbe o relaciji tog pristupa sa radovima pokojnog R. Stojanovića neka od najnovijih dostignuća u toj grani fizike kontinuuma su prikazana. Ona uključuju direktne posledice u numerici problema mehanike čvrstih tela, opšte primene na teorije plastičnosti, rasta i mešavina kao i nestandardnu primenu na progres fronta faznog prenosa u čvrstim telima.